

IB Math HL 2 Formal Calculus and Mathematical Induction Review ANSWERS

1. Given  $f(x) = \begin{cases} \sin(x-1) + cx & x \leq 1 \\ x^2 - x + d & x > 1 \end{cases}$ , find constants  $c, d \in \mathbb{R}$ . So that the function is differentiable at  $x = 1$ .

$$\sin(1-1) + c = 1^2 - 1 + d \quad \frac{d}{dx}(\sin(x-1) + cx) = \cos(x-1) + c$$

$$c = d \quad \frac{d}{dx}(x^2 - x + d) = 2x - 1$$

$$\cos 0 + c = 1$$

$$1 + c = 1$$

$$c = 0$$

$$\boxed{\begin{array}{l} c = 0 \\ d = 0 \end{array}}$$

2. Given  $f(x) = \begin{cases} -2x - 5 & x \leq -1 \\ x^2 - 4 & x > -1 \end{cases}$ , use Rolle's theorem to show that  $\frac{df}{dx} = 0$  exists on the interval  $[-2.5, 2]$ .

continuous?

$$-2(-1) - 5 = -3$$

$$(-1)^2 - 4 = -3 \quad \checkmark$$

Differentiable?

$$f'(x) = \begin{cases} -2 & x < -1 \\ 2x & x > -1 \end{cases}$$

$$-2 = 2(-1) \quad \checkmark$$

$$f(-2.5) = -2(-2.5) - 5 = 0$$

$$f(2) = 2^2 - 4 = 0$$

$f$  is differentiable on  $[-2.5, 2]$   
and  $f(-2.5) = f(2) = 0$ , so

$$\frac{df}{dx} = 0 \text{ for some } x \in [-2.5, 2]$$

3. Find  $\frac{df}{dx}$  where  $f(x) = \int_6^{\sin x} \frac{1}{\ln t} dt$ . Show your work with proper notations.

Suppose  $g(v)$  is an antiderivative of  $\frac{1}{\ln x}$ .

Then  $f(x) = g(\sin x) - g(6)$

$$\text{so } f'(x) = g'(\sin x)(\cos x) - 0$$

$$= \boxed{\frac{1}{\ln(\sin x)} \cdot \cos x}$$

4. Given  $f(x) = x + \frac{1}{x}$ , find all values of  $c$  on the given interval  $[1, 3]$  such that  $f(b) - f(a) = f'(c)(b - a)$ .

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f(3) - f(1) = f'(c)(3 - 1)$$

$$\frac{10}{3} - 2 = f'(c)(2)$$

$$\frac{2}{3} = f'(c)$$

$$-\sqrt{3} \notin [1, 3]$$

$$\frac{2}{3} = 1 - \frac{1}{c^2}$$

$$\text{so } \boxed{c = \sqrt{3}}$$

$$\frac{1}{c^2} = \frac{1}{3}$$

$$c^2 = 3$$

$$\rightarrow c = \pm \sqrt{3}$$

7. The function  $f$  is defined by  $f(x) = e^x \sin x$ .

a. Show that  $f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right)$ .

$$f'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$$

$$f''(x) = e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$$

$$= e^x (2 \cos x) = 2e^x \cos x = 2e^x \sin\left(x + \frac{\pi}{2}\right)$$

$\rightarrow$  trig identity

b. Obtain a similar expression for  $f^{(4)}(x)$ .

$$f'''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right) = 2e^x (\sin(x + \frac{\pi}{2}) + \cos(x + \frac{\pi}{2}))$$

$$f''''(x) = 2e^x (\sin(x + \frac{\pi}{2}) + \cos(x + \frac{\pi}{2})) + 2e^x (\cos(x + \frac{\pi}{2}) - \sin(x + \frac{\pi}{2}))$$

$$= 4e^x \cos(x + \frac{\pi}{2}) = 4e^x \sin((x + \frac{\pi}{2}) + \frac{\pi}{2})$$

$$= \boxed{4e^x \sin(x + \pi)}$$

OR  $2^2 e^x \sin(x + \frac{2\pi}{2})$

c. Suggest an expression for  $f^{(2n)}(x)$ ,  $n \in \mathbb{Z}^+$ , and prove your conjecture using mathematical induction.

$$f^{(2n)}(x) = 2^n e^x \sin\left(x + \frac{n\pi}{2}\right)$$

$$P(\underline{1}): f^{(2\underline{1})}(x) = 2e^x \sin(x + \frac{\pi}{2}) \quad P(\underline{1}) \text{ is true.}$$

Assume  $P(n)$  is true. Then  $f^{(2n)}(x) = 2^n e^x \sin\left(x + \frac{n\pi}{2}\right)$

$$P(n+1): f^{(2n+1)}(x) = 2^n e^x \sin\left(x + \frac{n\pi}{2}\right) + 2^n e^x \cos\left(x + \frac{n\pi}{2}\right)$$

$$= 2^n e^x (\sin(x + \frac{n\pi}{2}) + \cos(x + \frac{n\pi}{2}))$$

$$f^{(2n+2)}(x) = 2^n e^x (\sin(x + \frac{n\pi}{2}) + \cos(x + \frac{n\pi}{2}))$$

$$+ 2^n e^x (\cos(x + \frac{n\pi}{2}) - \sin(x + \frac{n\pi}{2}))$$

$$= 2 \cdot 2^n e^x \cos\left(x + \frac{n\pi}{2}\right)$$

$$= 2^{n+1} e^x \sin\left(x + \frac{n\pi}{2} + \frac{\pi}{2}\right)$$

$$= 2^{n+1} e^x \sin\left(x + \frac{(n+1)\pi}{2}\right)$$

so  $P(n+1)$  is true.  $P(\underline{1})$  is true. If  $P(n)$  is true then  $P(n+1)$  is true. Hence  $P(n)$  is true for all  $n \in \underline{\mathbb{Z}}^+$ .

8. (13G#3) From  $\sin 2x = 2 \sin x \cos x$  we observe that  $\sin x \cos x = \frac{\sin 2x}{2} = \frac{\sin(2^1 x)}{2^1}$ .

a. Prove that i.  $\sin x \cos x \cos 2x = \frac{\sin(2^2 x)}{2^2}$

$$\frac{\sin 2x}{2} \cdot \cos 2x$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \sin 2x \cos 2x = \frac{\sin 4x}{4}$$

ii.  $\sin x \cos x \cos 2x \cos 4x = \frac{\sin(2^3 x)}{2^3}$

$$\frac{\sin 2x}{2} \cdot \cos 2x$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \sin 4x \cos 4x = \frac{\sin 8x}{8}$$

b. Assuming the pattern in part a continues, simplify

i.  $\sin x \cos x \cos 2x \cos 4x \cos 8x$

$$\frac{\sin(2^4 x)}{2^4}$$

ii.  $\sin x \cos x \cos 2x \dots \cos 32x$

$$\frac{\sin(2^6 x)}{2^6 x}$$

c. i. Generalize the results from parts a and b.

$$\sin x \cos x \cos 2x \cos 4x \dots \cos(2^{n-1} x) = \frac{\sin(2^n x)}{2^n}, \quad n \in \mathbb{Z}^+$$

ii. Prove your generalization using mathematical induction.

$$P(1): \sin x \cos(2^0 x) = \frac{\sin 2^1 x}{2^1} \quad \checkmark$$

$P(1)$  is true.

Assume  $P(n)$  is true. Then

$$P(n+1): \sin x \cos x \cos 2x \cos 4x \dots \cos(2^n x)$$

$$= \sin x \cos x \cos 2x \cos 4x \dots \cos(2^{n-1} x) \cos(2^n x)$$

$$= \frac{\sin(2^n x)}{2^n} \cdot \cos(2^n x)$$

$$= \frac{1}{2^n} \cdot \frac{1}{2} \cdot 2 \sin(2^n x) \cos(2^n x)$$

$$= \frac{1}{2^{n+1}} \cdot \sin(2(2^n x))$$

$$= \frac{\sin(2^{n+1} x)}{2^{n+1}} \quad \checkmark$$

So  $P(n+1)$  is true.  $P(1)$  is true. If  $P(n)$  is true then  $P(n+1)$  is true. Hence  $P(n)$  is true for all  $n \in \mathbb{Z}^+$ .

9. Using Mathematical induction, prove  $\frac{d^n y}{dx^n} = (-1)^{n-1} \cdot \frac{2(n-3)!}{(1+x)^{n-2}}$  for  $n \in \mathbb{Z}, n > 2$  if  $y = (1+x)^2 \ln(1+x)$

P(3):

Assume  $P(n)$  is true.

P(n+1):

$$\begin{aligned}
 y &= (1+x)^2 \ln(1+x) \\
 \frac{dy}{dx} &= (1+x)^2 \cdot \frac{1}{1+x} + 2(1+x) \ln(1+x) \\
 &= (1+x) + 2(1+x) \ln(1+x) \\
 \frac{d^2y}{dx^2} &= 1 + 2(1+x) \cdot \frac{1}{1+x} + 2 \ln(1+x) = 3 + 2 \ln(1+x) \\
 \frac{d^3y}{dx^3} &= \frac{2}{1+x} \quad \checkmark \quad (-1)^{3-1} \cdot \frac{2(3-3)!}{(1+x)^{3-2}} = 1 \cdot \frac{2}{1+x} \quad \checkmark
 \end{aligned}$$

$P(3)$  is true

$$\begin{aligned}
 \frac{d^{n+1}y}{dx^{n+1}} &= \frac{d}{dx} \left( \frac{d^n y}{dx^n} \right) \\
 &= \frac{d}{dx} \left( (-1)^{n-1} \cdot \frac{2(n-3)!}{(1+x)^{n-2}} \right) \\
 &= (-1)^{n-1} \cdot 2(n-3)! \cdot (-n-2) \cdot \frac{1}{(1+x)^{n-1}} \\
 &= (-1)^{n-1} \cdot (-1) \cdot 2(n-3)! \cdot (n-2) \cdot \frac{1}{(1+x)^{n-1}} \\
 &= (-1)^n \cdot \frac{2(n-2)!}{(1+x)^{n-1}} \\
 &= (-1)^{(n+1)-1} \cdot \frac{2((n+1)-3)!}{(1+x)^{(n+1)-2}}
 \end{aligned}$$

so  $P(n+1)$  is true.

$P(3)$  is true. If  $P(n)$  is true then  $P(n+1)$  is true. Hence  $P(n)$  is true for all  $n \in \mathbb{Z}, n > 2$ .