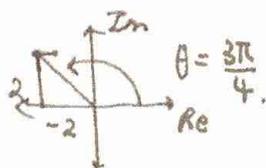


#1. $z^3 = -2 + 2i$



a) $z = (-2 + 2i)^{\frac{1}{3}}$

$$r = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$z = \left(\sqrt{8} \operatorname{cis} \left(\frac{3\pi}{4} + 2n\pi \right) \right)^{\frac{1}{3}}$$

$$z = 2^{\frac{2}{3} \cdot \frac{1}{3}} \operatorname{cis} \frac{1}{3} \left(\frac{3\pi}{4} + 2n\pi \right)$$

$n=0$ $z = \boxed{\sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right)} \Rightarrow \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$

$n=1$ $z = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} + \frac{2\pi}{3} \right) \rightarrow \boxed{\sqrt{2} \operatorname{cis} \frac{11\pi}{12}}$

$n=2$ $z = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} + \frac{4\pi}{3} \right) \rightarrow \boxed{\sqrt{2} \operatorname{cis} \frac{19\pi}{12}}$

b) When $n=0$ $z = \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \boxed{1+i}$

#2. $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$

$$\Rightarrow \sqrt{x^2 + y^2} + x + \boxed{yi} = 6 \boxed{2i} \Rightarrow \boxed{y = -2}$$

$$\sqrt{x^2 + 4} + x = 6$$

$$\left(\sqrt{x^2 + 4} \right)^2 = (6 - x)^2$$

$$x^2 + 4 = 36 - 12x + x^2$$

$$12x = 32$$

$$x = \frac{32}{12} = \boxed{\frac{8}{3}}$$

#3. $\frac{z}{z+2} = 2-i$

$$z = (2-i)(z+2)$$

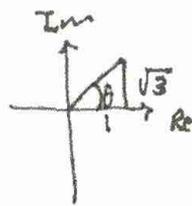
$$z = 2z + 4 - zi - 2i$$

$$zi - z = 4 - 2i$$

$$z(-1+i) = 4 - 2i$$

$$z = \frac{(4-2i)(-1-i)}{(-1+i)(-1-i)} = \frac{-4 - 4i + 2i - 2}{2} = \frac{-6 - 2i}{2} = \boxed{-3-i}$$

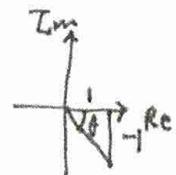
#4. $z_1 = a + a\sqrt{3}i = a(1 + \sqrt{3}i)$
 $= 2a \operatorname{Cis}\left(\frac{\pi}{3}\right)$



$$\theta = \frac{\pi}{3}$$

$$r = \sqrt{a^2 + 3a^2} = \sqrt{4a^2} = 2a$$

$$z_2 = 1 - i = \sqrt{2} \operatorname{Cis}\left(-\frac{\pi}{4}\right)$$



$$\theta = -\frac{\pi}{4}$$

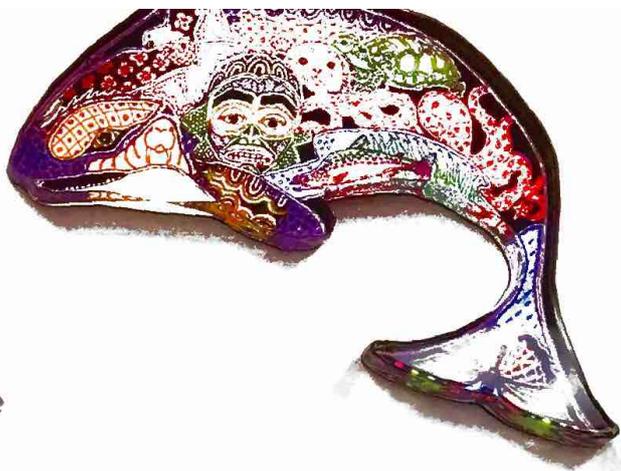
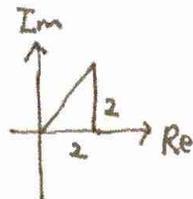
$$r = \sqrt{2}$$

$$\begin{aligned} \Rightarrow \left(\frac{z_1}{z_2}\right)^6 &= \left(\frac{2a \operatorname{Cis}\left(\frac{\pi}{3}\right)}{\sqrt{2} \operatorname{Cis}\left(-\frac{\pi}{4}\right)}\right)^6 = \left(\sqrt{2}a \operatorname{Cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right)^6 \\ &= (\sqrt{2}a)^6 \operatorname{Cis}\left[6\left(\frac{7\pi}{12}\right)\right] \\ &= 8a^6 \operatorname{Cis}\left[\frac{7\pi}{2}\right] \\ &= 8a^6 \left[\cos\left(\frac{7\pi}{2}\right) + i \sin\left(\frac{7\pi}{2}\right)\right] \\ &= 8a^6 \left[\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right] \\ &= 8a^6 [0 + i(-1)] = \boxed{-8a^6 i} \end{aligned}$$

#5 . a) $W = 2 + 2i$

$$|W| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\text{Arg}(W) = \frac{\pi}{4}$$



3

b) $z = \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) = \text{cis}\left(\frac{5\pi}{6}\right) \Rightarrow z^6 = \text{cis} 6\left(\frac{5\pi}{6}\right) = \text{cis}(5\pi)$

$$W = 2\sqrt{2} \text{cis}\left(\frac{\pi}{4}\right) \Rightarrow W^4 = (\sqrt{8})^4 \text{cis}\left(4\left(\frac{\pi}{4}\right)\right)$$

$$= 2^{\frac{3}{2} \cdot 4^2} \text{cis}(\pi) = 2^6 \text{cis}(\pi)$$

$$\Rightarrow W^4 \cdot z^6 = 2^6 \text{cis}(\pi) \cdot \text{cis}(5\pi) = 2^6 \text{cis}(6\pi) =$$

$$= 2^6 [\cos 6\pi + i \sin 6\pi] = 2^6 = \boxed{64}$$

#6. $u = 2 + 3i$ $v = 3 + 2i$

(a) $\frac{1}{u} + \frac{1}{v} = \frac{10}{W}$

$$\frac{1}{u} = \frac{(2-3i)}{(2+3i)(2-3i)} = \frac{2-3i}{4+9} = \frac{2-3i}{13}$$

$$\frac{1}{v} = \frac{(3-2i)}{(3+2i)(3-2i)} = \frac{3-2i}{4+9} = \frac{3-2i}{13}$$

$$\frac{1}{u} + \frac{1}{v} \Rightarrow \frac{2-3i}{13} + \frac{3-2i}{13} = \frac{5-5i}{13} = \frac{10}{W} \Rightarrow W \Rightarrow \frac{130}{5-5i} = \frac{26}{1-i} = \frac{26(1+i)}{13+13i}$$

(c) $16 \sin^4 \theta - 20 \sin^2 \theta + 5 = ?$



$$\begin{aligned} \text{a(i)} \Rightarrow \sin 5\theta &= 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta \\ &= \sin \theta [5(1 - \sin^2 \theta)^2 - 10(1 - \sin^2 \theta) \sin^2 \theta + \sin^4 \theta] \\ &= \sin \theta [5(1 - 2\sin^2 \theta + \sin^4 \theta) - 10\sin^2 \theta + 10\sin^4 \theta + \sin^4 \theta] \\ \sin(5\theta) &= \sin \theta [16 \sin^4 \theta - 20 \sin^2 \theta + 5] \end{aligned}$$

When $\theta = 72^\circ \Rightarrow \sin(5 \times 72^\circ) = \sin 360^\circ = 0$ where $\sin 72^\circ \neq 0$.

$$\therefore 0 = [16 \sin^4 \theta - 20 \sin^2 \theta + 5]$$

(d) $16 \sin^4 \theta - 20 \sin^2 \theta + 5 = 0$

$$\text{Q.F.} \Rightarrow \sin^2 \theta = \frac{20 \pm \sqrt{400 - (4)(16)(5)}}{2 \cdot 16} = \frac{20 \pm \sqrt{80}}{32} = \frac{5 \pm \sqrt{5}}{8}$$

$$\sin \theta = \pm \sqrt{\frac{5 \pm \sqrt{5}}{8}}$$

$$\text{When } \theta = 72^\circ \Rightarrow \therefore \sin \theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}} = \frac{\sqrt{5 \pm \sqrt{5}} \cdot \sqrt{2}}{2\sqrt{2} \cdot \sqrt{2}}$$

$$= \frac{\sqrt{10 \pm 2\sqrt{5}}}{4}$$

$(a=10, b=2, c=5, \text{ and } d=4)$

#7

(4)

(i)

$$(i) (\cos \theta + i \sin \theta)^5$$

$$= \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - i 10 \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$$

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 3 & 3 & 1 \\ & & & & & & 1 & 4 & 6 & 4 & 1 \\ & & & & & & 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

$$\Rightarrow \left[\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta \right] + i \left[5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta \right]$$

(ii) De Moivre's theorem

$$(\cos \theta + i \sin \theta)^5 = (1)^5 [\text{Cis } 5(\theta + 2n\pi)]$$

$$= \cos(5\theta + 10n\pi) + i \sin(5\theta + 10n\pi)$$

Equality.

$$(iii) \therefore \cos(5\theta) = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$

$$\therefore \sin(5\theta) = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$$

$$(b) z = r(\cos \phi + i \sin \phi) = r \text{Cis } \phi.$$

$$z^5 - 1 = 0 \Rightarrow z = (1)^{\frac{1}{5}} \quad 1 = \text{Cis}(360n)$$

$$= [\text{Cis}(360n)]^{\frac{1}{5}}$$

$$= [\text{Cis} \frac{1}{5}(360n)] = \text{Cis}(72n) \quad \text{where } n \in \mathbb{Z}$$

$$\phi = 72n \quad r = 1$$